

Cauchy's theorem and generalization

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Received: 15 December 2017 / Received in final form: 23 April 2018 / Accepted: 26 April 2018

Abstract. It has already been established that the mean length travelled by a neutral particle in a body containing a diffusing but not absorbing material is independent of its cross section, and consequently equal to the mean chord of the body. An elegant demonstration of this curious feature is presented and analysed thanks to Monte-Carlo simulations.

All the students in neutronics know the Cauchy's¹ theorem somehow related to the first collision probability theory. According to this theorem, the mean chord of a convex body is given by the very simple formula

$$\bar{X} = \frac{4V}{S}, \quad (1)$$

where V is the volume and S the surface of this body. The entry point A has to be chosen uniformly on the area of the surface and the inward direction isotropically. If B denotes the exit point, the chord X is the length of the segment AB .

This theorem, also known as the "theorem of the mean chord" or the "theorem of Dirac-Fuchs", was developed in 1943 by Paul Dirac and Klaus Fuchs, then working for the Manhattan Project.

It only applies for convex bodies in three-dimensional Euclidean geometry. The mean chord of a piece of fissile material is the key parameter for the functioning of an atomic bomb. For instance, the mean chord of a sphere is four thirds of its radius, and the one of a cube is two thirds of its edge length.

Generally this theorem is established by performing the integrations on the point A and the inward direction [1, Sect. 14.7, 2]. But a much more direct and simple neutronics² reasoning is also possible [1, Sect. 3.7].

¹ Augustin Cauchy, 1789–1857, French mathematician, was member of the Académie des Sciences and professor at the École polytechnique. He is the author of almost 800 articles and seven books.

² If you prefer, you can replace the neutrons with any neutral particles which do not interfere each other: photons, neutrinos, atoms of a very deluted gas...

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Let us imagine the whole space completely empty but a stationary, uniform and isotropic flux Φ of monokinetic particles.^{3,4}

– The number of particles continuously being in V is

$$N = Vn = \frac{V\Phi}{v}. \quad (2)$$

– For an isotropic flux the number of particles crossing per unit of time and unit of surface element is

$$J = \frac{\Phi}{4}. \quad (3)$$

– Thus the number of particles entering (and leaving) per unit of time through S is

$$E = JS = \frac{S\Phi}{4}. \quad (4)$$

– The mean life of a particle inside V is

$$\bar{t} = \frac{N}{E} = \frac{4V}{vS}. \quad (5)$$

– This mean life is also

$$\bar{t} = \frac{\bar{X}}{v}. \quad (6)$$

³ We use the term "flux" with its meaning in neutronics: the flux $\Phi = nv$ is the density n (mean number of particles per unit volume) multiplied by the velocity v .

⁴ The cosmic microwave background is an example of such a flux.

– Comparing these both expressions, we get the mean chord

$$\bar{X} = \frac{4V}{S}. \quad (7)$$

An extension of this theorem, established in particular in [2], was noticed in a large audience newspaper [3] as a curious feature concerning not the neutrons but the photons crossing water or scattered in milk. Here we will consider neutrons but that does not change the reasoning.

Without changing the stationary, uniform and isotropically external flux, let us now put inside V a homogeneous diffusing but not absorbing material. Nothing is changed concerning the entering neutrons. In order to know what happens inside V , let us write the Boltzmann's equation for this situation [1].

$$-\text{div}[\vec{\Omega}\Phi(\vec{r}, \vec{\Omega})] - \Sigma\Phi(\vec{r}, \vec{\Omega}) + \int_{(4\pi)} \Sigma_s(\vec{\Omega}' \rightarrow \vec{\Omega})\Phi(\vec{r}, \vec{\Omega}')d^2\Omega' = 0, \quad (8)$$

where \vec{r} is the space variable, $\vec{\Omega}$ the unit vector along the neutron velocity and $\Phi(\vec{r}, \vec{\Omega})$ the phase flux. Recall that the ordinary flux is linked to the phase flux by

$$\Phi(\vec{r}) = \int_{(4\pi)} \Phi(\vec{r}, \vec{\Omega})d^2\Omega. \quad (9)$$

Note that here the total cross-section equals the scattering one:

$$\Sigma = \Sigma_s, \quad (10)$$

(no absorption, only scattering) and that

$$\int_{(4\pi)} \Sigma_s(\vec{\Omega}' \rightarrow \vec{\Omega})d^2\Omega = \Sigma_s \forall \vec{\Omega}'. \quad (11)$$

It appears quite obviously that a flux constant in space satisfies this equation. As the flux must be a continuous function, it is the same inside and outside the body, and the same whichever this body is empty or contains a material.

Remark that the flux in an infinitesimal volume or integrated over a finite volume is the sum of the distances covered by all the particles in this infinitesimal or finite volume during a unit of time. Consequently, the sum of particle flight lengths remains the same with and without matter in V . As the number of entries also remains the same, this means that the average distance crossed by the particles through the body remains equal to \bar{X} if we introduce a diffusion material, whichever its cross section.

As the particles are assumed to be monokinetic, we can also say that the mean duration between the entry in the body and the exit from it is independant of the scattering cross section of the material it contains.

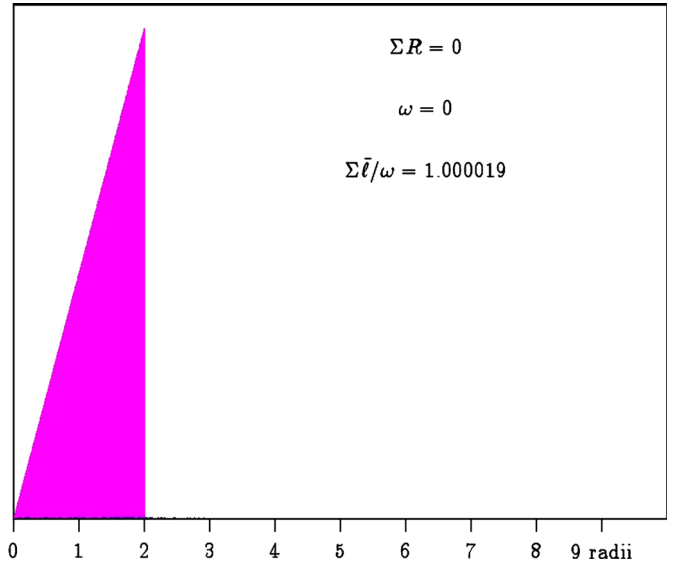


Fig. 1. Distribution of the lengths travelled by the particles for an empty sphere.

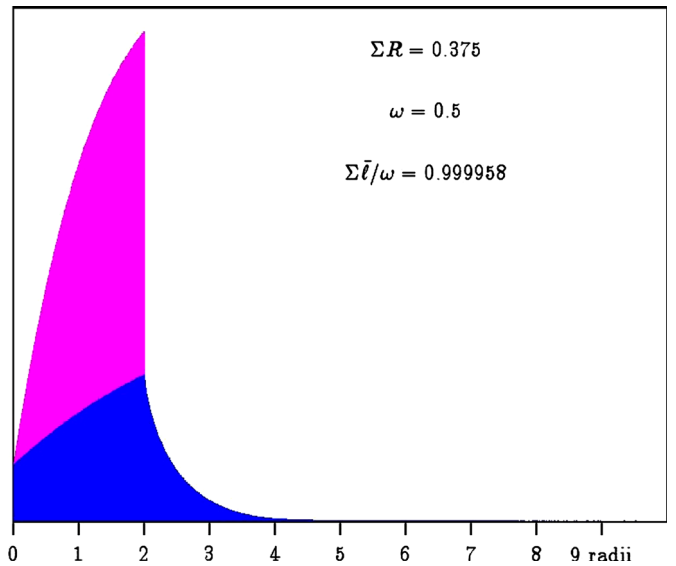


Fig. 2. Distribution of the lengths travelled by the particles for a sphere of opacity 0.5.

This cross section only affect the mean number of collisions per unit of time

$$C = V\Sigma\Phi. \quad (12)$$

Dividing this number by the mean number of entries per unit of time, equation (4), we get the mean number \bar{m} of collisions undergone by a particle between its entry and its exit

$$\bar{m} = \frac{C}{E} = \frac{4V\Sigma}{S} = \Sigma\bar{X}. \quad (13)$$

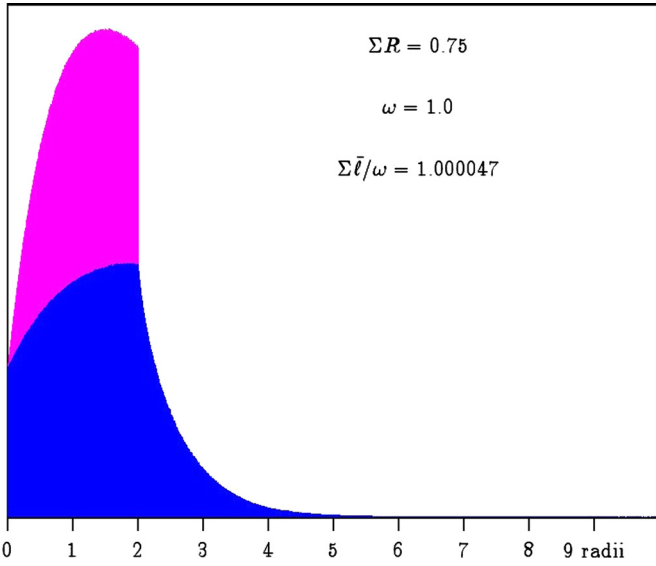


Fig. 3. Distribution of the lengths travelled by the particles for a sphere of opacity 1.

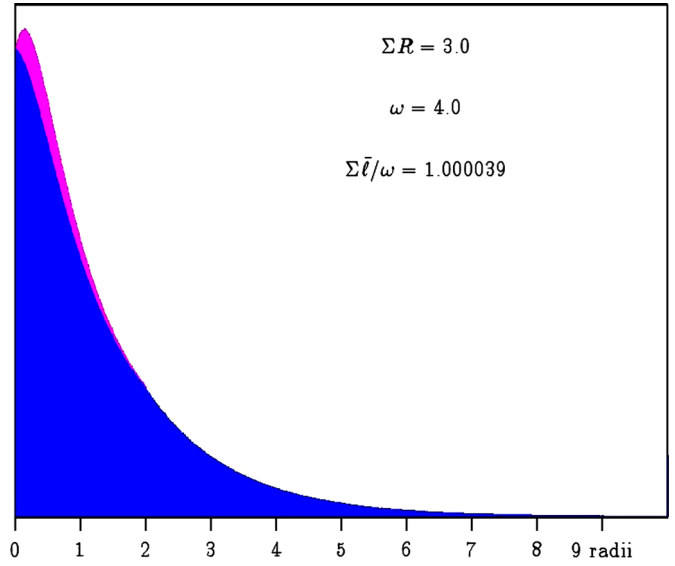


Fig. 5. Distribution of the lengths travelled by the particles for a sphere of opacity 4.

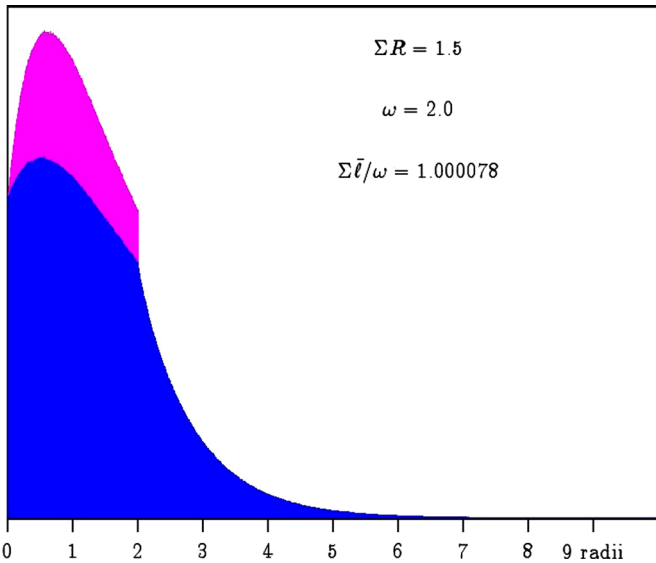


Fig. 4. Distribution of the lengths travelled by the particles for a sphere of opacity 2.

This parameter is called the “opacity” of the body [1, Sect. 3.1.5].

Figures 1–5 show, for five examples, that the greater the cross section of the material, the more frequent are the short and long “travels”. If the cross section is great, many particles undergo collisions near the surface and then leave after a short travel; but the particles which reach the central parts perform long travels. If the body is void, most of the travels are medium.

Appendix

In order to examine not only the mean value of the lengths travelled by the particles but also their probability distribution, we chose the simplest possible geometry – a sphere – and performed a Monte-Carlo simulation of the walks.

We simulated 10^8 particle walks and distributed the obtained lengths among 1500 classes of same width between 0 and 10 times the radius of the sphere.

We distinguished the paths without collision between the entry and the exit – which are presented in magenta – from the paths with at least one collision – which are presented in blue. The path without collision cannot be greater than the sphere diameter; consequently, the “magenta” curves stop at two radii. Of course, if the sphere is void, only “magenta” paths appear. It can be noted that the more opaque is the sphere, the rarer are the direct paths without collision.

A greater cross-section increases both the short paths (exit near the entry point after a small number of collisions) and the long ones (many collisions) without changing their mean value.

In the right upper part of the frame of the figures, three pieces of information are given.

- the first one is the radius of the sphere in unit “mean free path in the material”, that is to say the (scattering) macroscopic cross section of the material Σ times the radius R ;
- the second one is the opacity $\omega = \Sigma \bar{X}$. Note that $\omega = \frac{4}{3} \Sigma R$ for a sphere;
- the last information is the mean length travelled by the particles, Σ , evaluated by the Monte-Carlo simulation

and expressed in “mean free paths in the material”, compared with its theoretical value ω . The small discrepancies which can be observed denote the statistical errors of the Monte-Carlo simulations.

References

1. P. Reuss, *Neutron Physics* (EDP Sciences, Coll. Nuclear Engineering, 2008)
2. A. Mazzolo, C. de Mulatier, A. Zoia, *J. Math. Phys.* **55**, 083308 (2014)
3. D. Larousserie, *Comment la lumière se joue des obstacles. Les photons parcourent la même distance, qu'ils traversent de l'eau ou du lait*, *Le Monde, Sciences et médecine*, 2017
4. W.J.M. De Kruifj, J.L. Kloosterman, *Ann. Nuclear Energy* **30**, 549 (2003)
5. N.G. Sjöstrand, On the average chord length, in *Reactor Physics Calculations in the Nordic Countries* (Helsinki, 2003)
6. R. Sanchez, *Ann. Nuclear Energy* **31**, 2211 (2004)

Further reading

Cite this article as: Paul Reuss, Cauchy's theorem and generalization, EPJ Nuclear Sci. Technol. **4**, 50 (2018)